RECURSIVE NUMBER THEORY

A DEVELOPMENT OF RECURSIVE ARITHMETIC IN A LOGIC-FREE EQUATION CALCULUS

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1957

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM

INTRODUCTION

The Nature of Numbers

The question "What is the nature of a mathematical entity?" is one which has interested thinkers for over two thousand years and has proved to be very difficult to answer. Even the first and foremost of these entities, the natural number, has the elusiveness of a will-of-the-wisp when we try to define it.

One of the sources of the difficulty in saying what numbers are is that there is nothing to which we can point in the world around us when we are looking for a definition of number. The number seven, for instance, is not any particular collection of seven objects, since if it were, then no other collection could be said to have seven members; for if we identify the property of being seven with the property of being a *particular* collection, then being seven is a property which no *other* collection can have. A more reasonable attempt at defining the number seven would be to say that the property of being seven is the property which *all* collections of seven objects have in common. The difficulty about this definition, however, is to say just what it is that all collections of seven objects really do have in common (even if we pretend that we can ever become acquainted with *all* collections of seven objects). Certainly the number of a collection is not a property of it in the sense that the colour of a door is a property of the door, for we can change the colour of a door but we cannot change the number of a collection without changing the collection itself. It makes perfectly good sense to say that a door which was formerly red, and is now green, is the same door, but it is nonsense to say of some collection of seven beads that it is the same collection as a collection of eight beads. If the number of a collection is a property of a collection then it is a *defining* property of the collection, an essential characteristic.

This, however, brings us no nearer to an answer to our question

"What is it that all collections of seven objects have in common?" A good way of making progress with a question of this kind is to ask ourselves "How do we know that a collection has seven members?" because the answer to this question should certainly bring to light something which collections of seven objects share in common. An obvious answer is that we find out the number of a collection by *counting* the collection but this answer does not seem to help us because, when we count a collection, we appear to do no more than 'label' each member of the collection with a number. (Think of a line of soldiers numbering off.) It clearly does not provide a definition of number to say that number is a property of a collection which is found by assigning numbers to the members of the collection.

The Frege-Russell Definition

To label each member of a collection with a number, as we seem to do in counting, is in effect to set up a correspondence between the members of two collections, the objects to be counted and the natural numbers. In counting, for example, a collection of seven objects, we set up a correspondence between the objects counted and the numbers from one to seven. Each object is assigned a unique number and each number (from one to seven) is assigned to some object of the collection. If we say that two collections are *similar* when each has a unique associate in the other, then counting a collection may be said to determine a collection of numbers similar to the collection counted. Since similarity is a transitive property, that is to say, two collections are similar if each of them is similar to a third, it follows that in similarity we may have found the property, common to all collections of the same number, for which we have been looking, and since similarity itself is defined without reference to number it is certainly eligible to serve in a definition of number. To complete the definition we need only to specify certain standard collections of numbers one, two, three, and so on; a collection is then said to have a certain number only if it is similar to the standard collection of that number. The numbers themselves may be made to provide the required standards

in the following way. We define the property of being an empty collection as the property of not being identical with oneself, and then the number zero is defined as the property of being similar to the empty collection. Next we define the standard unit collection as the collection whose only member is the number zero, and the number one is defined as the property of being similar to the unit collection. Then the standard pair is taken to be the collection whose members are the numbers zero and unity and the number two is defined as the property of being similar to the standard pair, and so on. This is, in effect, the definition of number which was discovered by Frege in 1884 and, independently, by Russell in 1904. It cannot, however, be accepted as a complete answer to the problem of the nature of numbers. According to the definition, number is a similarity relation between collections in which each element of one collection is made to *correspond* to a certain element of the other, and vice-versa. The weakness in the definition lies in this notion of *correspondence*. How do we know when two elements correspond? The cups and saucers in a collection of cups standing in their saucers have an obvious correspondence, but what is the correspondence between, say, the planets and the Muses? It is no use saying that even if there is no patent correspondence between the planets and the Muses, we can easily establish one, for how do we know this, and, what is more important, what sort of correspondence do we allow? In defining number in terms of similarity we have merely replaced the elusive concept of number by the equally elusive concept of correspondence.

Number and numeral

Some mathematicians have attempted to escape the difficulty in defining numbers, by identifying numbers with numerals. The number one is identified with the numeral 1, the number two with the numeral 11, the number three with 111, and so on. But this attempt fails as soon as one perceives that the properties of numerals are not the properties of numbers. Numerals may be blue or red, printed or handwritten, lost and found, but it makes no sense to ascribe these properties to numbers, and, conversely, numbers may be even or odd, prime or composite but these are not properties of numerals. A more sophisticated version of this attempt to define numbers in terms of numerals, makes numbers, not the same thing as, but the *names* of the numerals; this escapes the absurdities which arise in attempting to *identify* number and numeral but it leads to the equally absurd conclusion that some one *notation* is the quintessence of number. For if numbers are the names of numerals then we must decide which numerals they name; we cannot accept the number ten for instance as both the name of the roman numeral and the arabic numeral. And if it is said that the number ten is the name of all the numerals ten then we reach the absurd conclusion that the meaning of a number word changes with each notational innovation.

The antithesis of "number" and "numeral" is one which is common in language, and perhaps its most familiar instance is to be found in the pair of terms "proposition" and "sentence". The sentence is some physical representation of the proposition, but cannot be identified with the proposition since different sentences (in different languages, for instance) may express the same proposition. If, however, we attempt to say just what it is that the sentences express we find that the concept of proposition is just as difficult to characterise as the concept of number. It is sometimes held that the proposition is something in our minds, by contrast with the sentence, which belongs to the external world, but if this means that a proposition is some sort of mental image then it is just another instance of the confusion of a proposition with a sentence, for whatever may be in our minds, whether it be a thought in words, or a picture, or even some more or less amorphous sensation, is a *representation* of the proposition, differing from the written or spoken word only because it is not a communication. In the same way we see that the view that number is indefinable, being something which we know by our intuition, again confuses number with numeral, that is confuses number with one of its representations.

Arithmetic and the Game of Chess

The game of chess, as has often been observed, affords an excellent parallel with mathematics (or, for that matter, with language itself). To the numerals correspond the chess pieces, and to the operations of arithmetic, the moves of the game. But the parallel is even closer than this, for to the problem of defining number corresponds the problem of defining the entities of the game. If we ask ourselves the question "What is the king of chess?" we find precisely the same difficulties arise in trying to find an answer which we met in our consideration of the problem of defining the concept of number. Certainly the king of chess, whose moves the rules of the game prescribe, is not the piece of characteristic shape which we *call* the king, just as a numeral is not a number, since any other object, a matchstick or a piece of coal, would serve as well to play the king in any game. Instead of the question "What is the king of chess?" let us ask "What makes a particular piece in the game the king piece?" Clearly it is not the shape of the piece or its size, since either of these can be changed at will. What constitute a piece king are its *moves*. That piece is king which has the king's moves. And the king of chess itself? The king of chess is simply one of the *parts* which the pieces play in a game of chess, just as King Lear is a part in a drama of Shakespeare's; the actor who plays the King is King in virtue of the *part* which he takes, the sentences he speaks and the actions he makes, (and not simply because he is dressed as king) and the piece on the chess board which plays the king-role in the game is the piece which makes the king's moves.

Here at last we find the answer to the problem of the nature of numbers. We see, first, that for an understanding of the meaning of numbers we must look to the 'game' which numbers play, that is to arithmetic. The numbers, one, two, three, and so on, are characters in the game of arithmetic, the pieces which play these characters are the numerals and what makes a sign the numeral of a particular number is the part which it plays, or as we may say in a form of words more suitable to the context, what constitute a sign the sign of a particular number are the *transformation rules* of the sign. It follows, therefore, that the OBJECT OF OUR STUDY IS NOT NUMBER ITSELF BUT THE TRANSFORMATION RULES OF THE NUMBER SIGNS, and in the chapters which follow we shall have no further occasion to refer to the number concept. But just as the rules of chess are currently formulated in terms of the entities of chess, so that we say, for instance, the king of chess moves only one square at a time (except in castling), instead of the completely equivalent formulation "the piece playing the part of king (or simply the king-piece) is moved only one square at a time (except in castling)" so we shall continue, in purely descriptive passages, to formulate the operations of arithmetic in terms of arithmetical entities instead of arithmetical signs. For instance, we may speak of "the sum of the numbers two and three" rather than confine ourselves to the object formulation "$2+3$", where $+$ is the sign whose *role* in arithmetic is what is called addition, and "2" and "3" are numerals whose roles are those of the numbers two and three. To put it another way, in defining the part played by a sign like $+$, in arithmetic, we shall say that what we are defining is the sum function, but the definition itself will refer only to operations for transforming *expressions* which contain the sign $+$.

Number Variables

The parallel between chess and arithmetic breaks down when we contrast the predetermined set of pieces in the game of chess with the licence granted to arithmetic to construct numerals at will. In this respect arithmetic more closely resembles a *language* which places no limit, in principle, upon the length of its words. A familiar notation for numerals expresses them as words spelt with the 'alphabet' "0", "1" and "+"; each 'word' has an initial "0" followed by a succession of pairs "+1". Thus, for instance, we form in turn "0", "$0+1$", "$0+1+1$", "$0+1+1+1$". The formation of numerals may be *fully* characterised by means of two operations, as follows. We extend the alphabet by the introduction of a new sign, "x", and form 'words' by writing either "0" or "$x+1$" for "x"; for example we may form in turn, "x", "$x+1$", "$x+1+1$", "$x+1+1+1$", "$0+1+1+1$", the last of which is a numeral. This new sign we call a 'numeral variable'. The rules permitting the substitution of "$x+1$" or "0" for "x" in effect allow the substitution of *any numeral* for x; the object of the formulation we have adopted is that it serves to define the concept of *any numeral* and the concept of a *numeral variable* simultaneously. In the sequel, not only the letter x, but other letters, too, will be used as numeral variables.

The numeral formed by writing some numeral for "x" in "$x+1$" is called the *successor* of that numeral. For instance, writing "$0+1+1$" for "x" in "$x+1$" we obtain "$0+1+1+1$", the successor of "$0+1+1$". For this reason "$x+1$" is called the (sign of the) *successor function*. The definite article is somewhat misleading, however, since we may write, in place of x, any other letter which is being used as a numeral variable; in a system in which x, y and z are all numeral variables, each of "$x+1$", "$y+1$", "$z+1$" is a sign of the successor function. Nevertheless we shall talk of *the* successor function, the uniqueness in question being the uniqueness of the sign which results when we write some definite numeral for the variable, be it denoted by x, y or z.

For purposes of standardisation of notation we shall have occasion to introduce, instead of the 'alphabet' "0", "1" and "+" for writing numerals, the 'alphabet' "0", "S" in which the numerals become "0", "$S0$", "$SS0$", "$SSS0$" and so on. In this notation the sign of the successor function is "Sx" and the transformation rules for a numeral variable x are (i) Sx may be written for x, (ii) 0 may be written for x.

Another notation in current use employs "x'" for the successor function, so that the numerals are written "0", "$0'$", "$0''$", "$0'''$" and so on.

Definition of Counting

No theory of the natural numbers is complete which does not also take into account the part which numbers play *outside* arithmetic. It is not only a property of the number nine that it is a square but also that it is the number of the planets, and this latter property is not a consequence simply of the laws of arithmetic. According to the Frege–Russell definition of number, the number of a collection is found by testing it for similarity with the standard unit, pair, trio, and so on, in turn, this testing being carried out by the process of counting, but as we have proposed a definition of number which does not rest upon the undefined concept of a similarity correspondence we cannot accept counting, in the Frege–Russell sense, as a means of finding the number of a class, without readmitting this undefined concept. There is, however, an entirely different interpretation of the process of counting, which makes counting available to us as a means of recording the number of a collection, without transcending the limitation we imposed upon ourselves of expressing the properties of numbers in terms of the transformation rules of the number signs. We start by separating two distinct stages in the process of counting. The first of these is what we shall call "using a collection as a numeral" which consists in overlooking the individual 'idiosyncrasies' of the elements of the collection and regarding them as being all alike (but not identical) for the purpose in hand. This is simply a (perhaps rather extreme) form of a treatment of signs familiar in all acts of reading, writing or speaking; the letters "*a*" on a printed page, for instance, have their several differences and, subject to sufficiently close scrutiny, are as different as say the soldiers in a platoon, but for the purposes of reading we ignore these differences and treat the various *a*'s as being the same sign. And so too, in speaking, we treat as the same a variety of slightly different sounds. In a different context, signs which we would accept as the same for reading purposes, are carefully distinguished, as, for example, when we test the quality of printing. The process of overlooking some differences, but not others, is fundamental in language; it is the process by which we subsume objects with a 'family likeness' under a generic name and the process which makes possible the use in language of universal words. Without it, the concept of the number of a class could never have arisen. The second stage in the process of counting consists in a transformation from one number notation to another by means of the rules "one and one is two", "two and one is three", "three and one is four" and so on. It is the recitation of these rules (in an abbreviated form in which each 'and one' is omitted, or replaced by pointing to, or touching, the object counted) which gives rise to the illusion that in counting we are associating a number with each of the elements counted, whereas we are in fact making a translation from the notation in which the number signs are "one", "one and one", "one and one and one", and so on, to the notation in which the signs are "one", "two", "three", and so on. The true nature of counting is perhaps most clearly brought out if we re-introduce the older process of making a tally. Making a tally of a collection consists in some formalised representation of the elements of the collection, say by means of dashes on a sheet of paper, so that in making a tally we are *copying* a number sign in some standard notation – finding the number of the collection, by treating it as a number sign and copying this sign. Thus a tally of the planets consists in the row of dashes

$$111111111$$

If we now proceed to transform this sign by means of the transformation rules $11=2$, $21=3$, $31=4$, $41=5$, $51=6$, $61=7$, $71=8$, $81=9$ we obtain in turn $111111111=21111111=3111111=411111=$ $=51111=6111=711=81=9$, which completes the transformation. In counting as we teach it today, the processes of tally making and sign transformation are carried out simultaneously, thus avoiding the repeated copying of the 'tail' of the number sign in transforming to an arabic numeral. It is important to realise that counting does not discover the number of a collection but *transforms the numeral which the collection itself instances* from one notation to another. To say that any collection has a number is just to say that any collection may be used as a number sign.

Formalisation of Counting

Counting may be formalised in a system of signs by formulating the transformation rules of a counting operator "N". We represent the objects in the collections to be counted by letters a, b, c, ..., and collections by conjunctions like $a \;\&\; b$, $a \;\&\; b \;\&\; c$; a single object being regarded also a collection. The letter l we use as a variable for an object, that is, a letter for which any object may be written; the capital letter L serves as a variable for a collection and may, in any context, be replaced by a definite collection or by "$L \;\&\; l$". The numerals of the system are the signs (without x) obtained from l, x and the successor function $x+1$ by substitution. Then we define

$$N(l)=1, \qquad N(L \;\&\; l)=N(L)+1.$$

These equations suffice to determine the number of any collection. For instance, substituting "a" for the variable-sign "l", in the first, we obtain $N(a)=1$, and then, substituting "a" for "L" and "b" for "l" in the second, we obtain

$$N(a \;\&\; b)=N(a)+1$$

and so, $N(a \;\&\; b)=1+1$.

Next, substituting "$a \;\&\; b$" for "L" and "c" for "l" we find

$$N(a \;\&\; b \;\&\; c)=N(a \;\&\; b)+1=1+1+1,$$

and so on.

We observe that the definition of $N(L)$ is *by recursion*, that is to say, $N(L)$ is not simply an abbreviation for some other expression, as, for instance, when we define $2=1+1$, the sign "2" may be replaced by "$1+1$" for which it is merely an abbreviation, but $N(L)$ is determined only *step by step*, by introducing the members of the class to be counted one at a time (or shedding them one at a time). We may express this by saying that for the variable L, $N(L)$ itself is undefind, only the result of substituting a definite class (like $a \;\&\; b \;\&\; c$) for L being defined by the recursive definition. The recursive definition is, so to speak, a *schema* or *mould* from which the definition (value) of $N(a \;\&\; b \;\&\; \ldots \;\&\; k)$ may be found by substitution for any particular class $a \;\&\; b \;\&\; \ldots \;\&\; k$.

Evolution of the Concept of a Formal System

In the following chapters we shall set up arithmetic as a *formal system*. The idea of a formal system is one which derives from Euclid's presentation of geometry, but the notion has undergone

considerable development during the past century. Euclid's intention in the "*Elements*" was to deduce the whole body of geometrical knowledge of his time from a few self evident truths (called axioms) by purely logical reasoning. Euclid did not, however, specify the nature of 'logical reasoning' and the first attempt to do so was made by George Boole, in 1847, in his *Mathematical Analysis of Logic*. Boole constructed a symbolic language, in which the 'laws of thought', formulated as axioms, may be studied by mathematical techniques. In the complete development of the notion a formal system is an assemblage of signs separated into various categories, their usage bound by various conventions (the axioms and transformation rules) the object of the system being to arrange sequences of formulae (which are themselves sequences of signs with certain specified formation rules) in certain relationships to one another to form a particular pattern called *proof*. A formal system may contain both mathematical and logical signs (the distinction is an arbitrary one), and mathematical and logical axioms; its essential feature, *qua* formal system, is that its operation does not presuppose any knowledge of the significance of the signs of the system than is given by the axioms and transformation rules. The mathematical axioms are no longer "self evident truths" but arbitrary initial positions in a game, and the logical axioms express, not the "laws of thought" but arbitrary conventions for the use of the logical signs.

In the formal system with which we shall first be concerned in this book, the equation calculus, the only signs are signs for functions and numeral variables, and the equality sign. There are no axioms except the introductory equations for function signs, and there is no appeal to 'logic', the operation of the system being specified simply by the transformation rules for the mathematical signs. It is shown that a certain branch of logic is *definable in the equation calculus* and logical signs, and theorems, are introduced as convenient *abbreviations* for certain functions and formulae. This branch of logic is characterised by the fact that it can assert the *existence* of a number with a given property only when the number in question can be found by a specifiable number of trials.